

# IDEAL GAS STEPHANI UNIVERSES

B. COLL

*Systèmes de référence spatio-temporels, CNRS  
DANOF, Observatoire de Paris  
F-75014 Paris, France.  
E-mail: bartolome.coll@obspm.fr*

J.J. FERRANDO

*Departament d'Astronomia i Astrofísica, Universitat de València,  
E-46100 Burjassot, València, Spain.  
E-mail: joan.ferrando@uv.es*

The Stephani Universes that can be interpreted as an ideal gas evolving in local thermal equilibrium are determined, and the method to obtain the associated thermodynamic schemes is given.

## 1 Introduction

The conformally flat perfect fluid solutions of Einstein equations with nonzero expansion were considered by Stephani<sup>1</sup> and are usually called Stephani Universes. These solutions were rediscovered by Barnes<sup>2</sup> who obtained them as the conformally flat class of irrotational and shear-free perfect fluid spacetimes with nonzero expansion.

In order to generalize the cosmological principle, Bona and Coll<sup>3</sup> looked for spacetimes admitting an iso-invariant synchronization, and without any hypothesis on the energy tensor, they likewise found the Stephani Universes.

An iso-invariant synchronization implies that an irrotational and shear-free observer  $u$  with non zero expansion exists, and the induced metric on the orthogonal hypersurfaces are of constant curvature. Then, there exists an adapted coordinate system,  $u = \frac{1}{\alpha}\partial_t$ , such that the line element writes:

$$ds^2 = -\alpha^2 dt^2 + \Omega^2 \delta_{ij} dx^i dx^j$$

$$\Omega \equiv \frac{R(t)}{1 + 2\vec{b}(t) \cdot \vec{r} + \frac{1}{4}K(t)r^2}, \quad \alpha \equiv R\partial_R \ln \Omega \quad (1)$$

$R(t)$ ,  $\vec{b}(t)$  and  $K(t)$  being 5 arbitrary functions of time.

The metric (1) is a perfect fluid solution,  $T = (\rho + p)u \otimes u + pg$ , with energy density and pressure given by

$$\rho = \frac{3\dot{R}^2}{R^2} + \frac{3}{R^2}(K - 4b^2), \quad p = -\rho - \frac{3}{R} \frac{\partial_R \rho}{\alpha} \quad (2)$$

Moreover, the expansion of  $u$  and the curvature of the spatial metric are homogeneous, and they are given, respectively, by

$$\theta(t) = \frac{3\dot{R}}{R} \neq 0, \quad \kappa(t) = \frac{1}{R^2}(K - 4b^2) \quad (3)$$

Stephani Universes are perfect fluid solutions of Einstein equations, but what is the physical meaning of these perfect fluids? Bona and Coll<sup>3</sup> gave the first step to answer this question: they studied when these perfect fluid solutions admit a thermodynamic scheme, and they showed that the thermodynamic Stephani Universes admit at least a three-dimensional group of isometries. This topic has been again considered later<sup>4</sup> and, in a recent paper, Sussman<sup>5</sup> analyzes a family of spherically symmetric Stephani Universes that may be interpreted as either a classical monoatomic ideal gas or a matter-radiation mixture.

Here we determine *all* Stephani Universes that represent the evolution in local thermal equilibrium of a (generic) ideal gas. In section 4 we present the main results that are based on previous ones: firstly, on the above cited work by Bona and Coll<sup>3</sup> about the thermodynamic schemes in this class of solutions, that we shorten here in section 2; secondly, on our hydrodynamic characterization of an ideal gas in local thermal equilibrium<sup>6</sup>, that we summarize in section 3.

## 2 Thermodynamic Stephani Universes

The usual physical interpretation of a perfect fluid follows when it has a conservative evolution in local thermal equilibrium (l.t.e.). This means:

- Energy-momentum conservation:  $\nabla \cdot T = 0$ .
- The energy density  $\rho$  is decomposed in terms of the matter density  $r$  and the specific internal energy  $\epsilon$ :  $\rho = r(1 + \epsilon)$ .
- The equation of conservation of matter is required:  $\nabla \cdot (ru) = 0$ .
- The thermodynamic variables are related by equations of state compatibles with the thermodynamic relation:  $Tds = d\epsilon + pd(1/r)$ .

Bona and Coll<sup>3</sup> looked for the Stephani Universes admitting the above thermodynamic scheme. The results that will be useful for us here can be summarized in the following two lemmas.

**Lemma 1** *A Stephani Universe admits a barotropic thermodynamic scheme iff it admits a six-dimensional isometry group.*

So, this case corresponds to the Friedmann-Robertson-Walker limit that occurs when  $K = \text{constant}$  and  $\vec{b} = \text{constant}$ .

**Lemma 2** *A Stephani Universe admits a strict (non barotropic) thermodynamic scheme iff it admits a three-dimensional isometry group.*

*For the thermodynamic Stephani Universes, the metric may be written*

$$\begin{aligned}
 ds^2 &= -\alpha^2 dt^2 + \Omega^2 \delta_{ij} dx^i dx^j \\
 \Omega &\equiv \frac{w}{z} L, & \alpha &\equiv R \partial_R \ln L \\
 L &\equiv \frac{R(t)}{1 + K(t)w}, & w &\equiv \frac{z}{1 + \frac{\epsilon}{4} r^2}
 \end{aligned} \tag{4}$$

$R(t)$  and  $K(t)$  being two arbitrary functions of time.

The symmetry group is spherical, plane or pseudospherical depending on  $\epsilon$  to be 1, 0 or  $-1$ .

For the thermodynamic perfect fluid solutions (4) the expressions (2) for the energy density and pressure become

$$\rho = \frac{3\dot{R}^2}{R^2} + \frac{3}{R^2}(\epsilon - K^2), \quad p = -\rho - \frac{3}{R} \frac{\partial_R \rho}{\alpha} \tag{5}$$

### 3 Ideal gas evolving in local thermal equilibrium

The definition of local thermal equilibrium given at the beginning of the previous section introduces thermodynamic variables, like  $r$ ,  $\epsilon$ ,  $s$ , and  $T$ , that are not present in the perfect fluid energy tensor. Then, a natural question arises: does an equivalent formulation of the l.t.e. exist that reduces to the addition to the energy-momentum conservation equation of a new condition involving only the hydrodynamic variables  $(u, \rho, p)$ ? We gave years ago<sup>7</sup> a positive answer to this question:

**Lemma 3** *A divergence-free perfect fluid energy tensor evolves in l.t.e. if, and only if, the space-time function  $\chi \equiv \dot{p}/\dot{\rho}$ , called the indicatrix of l.t.e., depends only on the variables  $p$  and  $\rho$  :  $d\chi \wedge dp \wedge d\rho = 0$*

This result has the conceptual interest of giving an exclusively hydrodynamic version of l.t.e.. But moreover it also has a practical utility because

provides a deductive algorithm to detect whether or not a given divergence-free perfect fluid energy tensor evolves in l.t.e.. In the Spanish Relativity Meeting-96<sup>8</sup> we pointed out these applications and gave a concrete example studying the thermodynamic class II Szekeres-Szafron space-times.

Nevertheless, in order to have a more accurate physical meaning of a perfect fluid it will be useful to study the indicatrix function of several particular media. Elsewhere<sup>6</sup> we have given this hydrodynamic characterization for some representative cases. We present below in two propositions a part of our results about ideal gases that we will use in next section.

A (generic) ideal gas satisfies the equation of state  $p = krT$ . For it we have the following hydrodynamic characterization<sup>6</sup>:

**Proposition 1** *The necessary and sufficient condition for a non barotropic and non isoenergetic ( $\dot{p} \neq 0$ ) divergence-free perfect fluid energy tensor  $(u, \rho, p)$  to represent the l.t.e. evolution of an ideal gas is that the indicatrix function  $\chi \equiv \dot{p}/\dot{p}$  be a function of the variable  $\pi \equiv p/\rho$ ,  $\chi = \chi(\pi) \neq \pi$ :*

$$d\chi \wedge d\pi = 0, \quad \chi \neq \pi \quad (6)$$

**Proposition 2** *A non barotropic and non isoenergetic divergence-free perfect fluid energy tensor  $(u, \rho, p)$  verifying (6) represents the l.t.e. evolution of the ideal gas with specific internal energy  $\epsilon$ , temperature  $T$ , matter density  $r$ , and specific entropy  $s$  given by*

$$\epsilon(\rho, p) = \epsilon(\pi) \equiv e(\pi) - 1; \quad T(\rho, p) = T(\pi) \equiv \frac{\pi}{k} e(\pi) \quad (7)$$

$$r(\rho, p) = \frac{\rho}{e(\pi)}; \quad s(\rho, p) = s_0 + k \left[ \int \phi(\pi) d\pi - \ln \rho \right] \quad (8)$$

$e(\pi)$  and  $\phi(\pi)$  being, respectively,

$$e(\pi) = e_0 e^{\int_0^\pi \psi(\pi) d\pi}, \quad \psi(\pi) \equiv \frac{\pi}{(\chi(\pi) - \pi)(\pi + 1)}; \quad \phi(\pi) \equiv \frac{1}{\chi(\pi) - \pi} \quad (9)$$

The above two propositions provide an algorithm to detect, in any given family of divergence-free perfect fluids  $\mathbf{T} = \{T \equiv [u^\alpha(x^\beta), \rho(x^\beta), p(x^\beta)]\}$ , those that represent an ideal gas evolving in local thermal equilibrium:

1. Calculate the coordinate dependence of the space-time functions  $\dot{p}/\dot{\rho} = \chi(x^\beta)$  and  $p/\rho = \pi(x^\beta)$
2. Determine the gas ideal subset of  $\mathbf{T}$  by imposing the hydrodynamic condition of the proposition 1:  $d\chi \wedge d\pi = 0$ .

3. Obtain the explicit expression of the indicatrix function:  $\chi = \chi(\pi)$ .
4. Calculate, from  $\chi(\pi)$ , the functions  $e(\pi)$  and  $\psi(\pi)$  given in (9), and obtain the thermodynamic variables using (7) and (8).

#### 4 Ideal gas Stephani solutions

In order to determine the ideal gas Stephani Universes we start from the thermodynamic ones (4), and we must study the restrictions that our ideal gas characterization imposes on the functions  $R(t)$  and  $K(t)$ . We follow the algorithm presented at the end of previous section:

##### 1. Variables $\chi$ and $\pi$

From (5) and (4), a direct calculation lends to:

$$p/\rho = \pi(R, w) = \frac{a(1 + Kw)}{1 + (K - RK')w} - 1, \quad a = a(R) \equiv -\frac{R\rho'}{3\rho} \quad (10)$$

$$\dot{p}/\dot{\rho} = \chi(\pi, R) = \pi + \frac{1}{3} - \frac{1}{3}(\pi + 1)\left[\frac{a'R}{a^2} + \frac{1}{a} + (\pi + 1 - a)\frac{RK''}{a^2K'}\right] \quad (11)$$

where the prime indicates derivative with respect to the variable  $R$ .

##### 2. Ideal gas hydrodynamic condition: $d\chi \wedge d\pi = 0$

This condition restricts, in a first step, the functions  $\rho(R)$  and  $K(R)$  that turn out to be submitted to the second order differential equations

$$a'' - \frac{2a'^2}{a} = a' \frac{K''}{K'} \quad (12)$$

$$K''' - K''\left(\frac{K''}{K'} - \frac{1}{R}\right) = 2K''\frac{a'}{a} \quad (13)$$

Secondly, we put every solution  $\rho(R)$ ,  $K(R)$  for these equations in the expression (5) of the energy density  $\rho$  and we obtain a generalized Friedmann equation for the expansion factor  $R(t)$ :

$$\rho(R) = \frac{3\dot{R}^2}{R^2} + \frac{3}{R^2}[\epsilon - K^2(R)] \quad (14)$$

### 3. Indicatrix function: $\chi = \chi(\pi)$

At this point we look for the functional expression to the indicatrix function. Assuming equations (12) and (13), the indicatrix (11) becomes

$$\chi(\pi) = \alpha\pi^2 + \gamma\pi + \beta, \quad \alpha + \beta = \gamma - 2/3 \quad (15)$$

### 4. Thermodynamic variables

Finally, the form (15) of the indicatrix function determines, using the results in proposition 2, the other thermodynamic variables, and so, the properties of the associated thermodynamic scheme.

In contrast to the Sussman<sup>5</sup> intuitive approach leading to very partial results, our algorithm gives directly, and in the above simple form, the *whole* class of ideal gas Stephani Universes. After that, we may look for a more specific ideal gases by imposing the corresponding restrictions to the indicatrix function (15). For example, if one collapses the whole space of solutions of equations (12) (13) to the particular solutions  $a(R) = \text{Constant}$ , and one imposes the gas to be monoatomic, one obtains the partial results by Sussmann. The physical meaning of the other cases will be considered elsewhere.

### Acknowledgments

This work has been supported by the Spanish DGES (Project PB96-0797).

### References

1. H. Stephani, *Comm. Math. Phys.* **4**, 137 (1967)
2. A. Barnes, *Gen. Rel. Grav.* **4**, 105 (1973).
3. C. Bona and B. Coll, *Gen. Rel. Grav.* **20**, 279 (1988)
4. A. Krasinski, H. Quevedo and R.A. Sussman, *J. Math. Phys.* **28**, 2602 (1997).
5. R.A. Sussman *Gen. Rel. Grav.* **32**, 1572 (2000).
6. B. Coll and J.J. Ferrando, *J. Math. Phys.* (to be submitted).
7. B. Coll and J.J. Ferrando, *J. Math. Phys.* **30**, 2918 (1989).
8. B. Coll and J.J. Ferrando, in *Some Topics on General Relativity and Gravitational Radiation*, ed. J.A. Miralles, J.A. Morales and D. Sáez (Ed. Frontières, Paris, 1997).